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BIFURCATION OF SINGULAR SOLUTIONS IN REVERSIBLE SYSTEMS AND APP--ETC(U)
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BIFURCATION OF SINGULAR SOLUTIONS IN REVERSIBLE SYSTEMS AND APPLICATIONS TO REACTION-DIFFUSION EQUATIONS

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ABSTRACT

Dynamical systems that are reversible in the sense of Moser are investigated and bifurcation of trajectories connecting saddle points from stationary solutions is studied. As an application, reaction-diffusion models in one space dimension are considered. These equations are studied in the neighborhood of a point, where the set of spatially homogeneous solutions displays a Hopf bifurcation. It is shown that from such a point branches of solutions bifurcate, which can be described as waves travelling to or from a center. These waves may be exponentially damped at infinity or not. They can be regarded as one-dimensional analogues of "target patterns" or "spiral waves".

AMS (MOS) Classifications: 34C35, 35B32, 35K55, 47H15, 80A30

Key Words: Bifurcation, Reversible Systems, Separatrices, Reaction-Diffusion Equations, Target Patterns, Spiral Waves

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SIGNIFICANCE AND EXPLANATION

In experiments on reaction-diffusion systems, e.g. the famous Belousov-Zhabotinskii reaction, patterns of rotating spirals or propagating concentric rings are observed. These patterns have found considerable theoretical interest in the recent literature. In this paper we give a rigorous proof for the existence of certain solutions to reaction-diffusion equations. The qualitative features of these solutions are such that they may be regarded as one-dimensional analogues of these patterns. Mathematically, the problem is a degenerate case for bifurcation leading to trajectories connecting critical points in a reversible system.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BIFURCATION OF SINGULAR SOLUTIONS IN REVERSIBLE SYSTEMS AND APPLICATIONS TO REACTION-DIFFUSION EQUATIONS

M. Renardy

0. INTRODUCTION

Recently Kirchgässner and Scheurle [19], [20] have constructed bounded nonperiodic solutions (which they call "singular") of reversible systems as "envelopes" of periodic solutions with infinitely increasing periods. Under appropriate hypotheses, they prove that there are branches of such solutions bifurcating from a stationary solution. In [33] I have developed a different approach to these solutions. In particular, it turned out that the singular solutions are in fact trajectories connecting saddle points. The present paper presents some extensions to the results of [33]. A global bifurcation theorem for singular solutions is shown. Moreover, applications to reaction-diffusion models are discussed. We obtain solutions, which may be considered one-dimensional analogues of the patterns of concentric rings or spiral waves observed e.g. in the Belousov-Zhabotinskii reaction.

In order to keep the paper essentially self-contained, the main results of [33] are reviewed here, the reader will, however, be referred to [33] for some of the proofs. The paper is organized in two parts: In the first (§§ 1-5) we deal with reversible systems on an "abstract" level, in the second part (§§ 6,7) we consider reaction-diffusion equations.

We study a differential equation of the form

$$u' = \frac{du}{dx} = A(\mu)u + B(\mu,u). \tag{0.1}$$

Here μ is a real parameter, and u lies in a Banach space Y. $A(\mu)$ is a linear operator in Y composed of a bounded part depending smoothly on μ and an (in general unbounded) operator A_0 satisfying certain semigroup conditions, and B is a smooth bounded operator from $\mathbb{R} \times Y$ into Y satisfying $\|B(\mu,u)\| = O(\|u\|^2)$. Our principal assumption is that (0.1) is

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reversible in the sense of Moser [26]. This means that there is a linear operator $R \in \mathcal{L}(Y)$ such that $R^2 = \mathrm{id}$, $A(\mu)R = -RA(\mu)$, and $B(\mu,Ru) = -RB(\mu,u)$. We assume that for $\mu < 0$ the spectrum of $A(\mu)$ has positive distance from the imaginary axis, whereas at $\mu = 0$ a pair of eigenvalues passes through 0 and becomes imaginary for $\mu > 0$. Under these conditions, Kirchgässner and Scheurle [19],[20] have proved that, for each μ in a positive neighborhood of 0, there exists a one-parameter family of periodic orbits centered at the origin. They also prove that, under certain additional assumptions, bounded nonperiodic solutions can be constructed as a limit of these periodic solutions, the convergence being uniform in bounded intervals. These limiting solutions are called "singular".

In [33] an alternative approach to these solutions was given. The existence proof is not based on approximation by periodic orbits; on the contrary, it is used that singular solutions are isolated in a suitable function space. Parts of the proof use ideas related to those employed in [9], [21] and [28]. A bifurcation parameter ε is introduced, leading to a reduced equation for $\varepsilon = 0$. This reduced equation is not the linearization of (0.1), but a nonlinear approximation of "Ginzburg-Landau" type [12]. For the reduced problem, singular solutions can be given explicitly. A refined version of the implicit function theorem is used to prove the existence of singular solutions for $\varepsilon \neq 0$. Whereas the nature of the singular solutions remains an open problem in the work of Kirchgässner and Scheurle, it becomes clear here that they are in fact trajectories connecting saddle points.

Under generic assumptions we have to distinguish two different cases, which can be shown to correspond to the two different reversibility conditions in [19]. In the first case a two-sided branch of stationary solutions emerges from the point $u = 0$, $\mu = 0$: For $\mu > 0$, the point $u = 0$, or respectively, for $\mu < 0$, the bifurcating fixed point are connected to themselves by a biasymptotic trajectory. Using a different method, this case was also investigated in [21]. In the second case a one-sided branch of stationary points bifurcates from 0, i.e. either for $\mu > 0$ or for $\mu < 0$ we have two new fixed points. It depends on the direction of the bifurcation whether these are saddle points or not. If they are, they are connected to each other by two trajectories. In both cases one of the solutions represented by the singular trajectory is symmetric w.r. to R, i.e. $Ru(-x) = u(x)$. These results are reviewed in §§ 1-4 of this paper. In § 5 we use degree theory to extend this local bifurcation theorem to a global result. It is shown that branches of singular solution can terminate only by one of the two following mechanisms:

1. The asymptotic stationary point loses the property of being a saddle.

2. A suitably defined norm of the solutions tends to infinity.

There are numerous examples for the occurence of singular solutions in physical problems. The research of Kirchgässner and Scheurle was motivated by the study of stationary solutions to the Bénard and Taylor problems. Further examples occur in classical mechanics, in the theory of water waves [9], in the theory of Josephson junctions and in nonlinear optics [3].

In the second part of this paper we consider applications to reaction-diffusion equations in one space dimension. These equations have the form

$$D_i u_i'' = f_i(\mu, u_1, \ldots, u_n) + \dot{u}_i \tag{0.2}$$

The $'$ stands for the derivative with respect to the space variable x, and the $\cdot$ stands for the derivative with respect to time t. The f_i are assumed to be smooth functions. We restrict our attention to solutions periodic in time. Putting $u_i' = v_i$, (0.2) can be rewritten as a system, which is reversible under either of the mappings

$$R : (u_i,v_i) \to (u_i,-v_i) \quad \text{or} \quad \hat{R} : (u_i(t),v_i(t)) \to (u_i(t+\tfrac{T}{2}), -v_i(t+\tfrac{T}{2}))$$

(T denotes the temporal period). We study solutions of (0.2) in the neighborhood of a point where the system

$$f_i(\mu,u_1,\ldots,u_n) + \dot{u}_i = 0 \tag{0.3}$$

undergoes a Hopf bifurcation (this has been shown to occur e.g. for the Brusselator [1], [12] and the Belousov-Zhabotinskii reaction [6], [13], [17], [18], [27], [31]). For (0.2) this leads to a situation, which has analogies to the one considered in the first part of the paper, but is more complicated; the eigenvalue 0 occuring in the linearized problem for $\mu = 0$ now has the algebraic multiplicity 4 and the geometric multiplicity 2 rather than 2 and 1 . Again we introduce a bifurcation parameter ε, and, taking into account only the terms of lowest order, we obtain a reduced equation.

This reduced equation turns out to be the simplest case of a "λ-ω-system". λ-ω-systems have been introduced as one theoretical concept to explain patterns of concentric rings ("target patterns") or rotating spirals ("spiral waves") occuring in chemical reactions (see e.g. [5], [8], [10], [11], [15], [22], [23], [29], [34], [35], [36], [38]). Solutions of λ-ω-systems were investigated in particular by Kopell and Howard [15], [22], [23] in one space dimension and by Greenberg [10], [11] in two space dimensions.

Our analysis focusses on two specific solutions, which we can give explicitly for $\varepsilon = 0$. Again a generalized implicit function theorem is used to prove persistence of solutions with the same qualitative properties for $\varepsilon \neq 0$. The first type of solutions are temporally periodic, symmetric with respect to R, and approach a constant as $x \to \pm\infty$, asymptotically they can for large $|x|$ be described as exponentially damped waves propogating in opposite directions for x positive and x negative. The solutions of the second type are temporally periodic, symmetric with respect to $\hat{R}$, in the limit $x \to \pm\infty$ they approach periodic wave trains, and the directions of propogation are again opposite.

The first kind of solutions can be regarded as a one-dimensional analogue of target patterns, whereas the second kind are one dimensional spiral waves [38]. In the case $\varepsilon = 0$, the latter solutions coincide with one of the solutions, for which existence was proved by Kopell and Howard in [23]. For $\varepsilon \neq 0$ these solutions were discussed on a formal level by Cohen, Hoppenstaedt and Miura [4].

It remains an open question in general whether the solutions under study here can be stable or not. In chapter 7 we investigate the stability of the solutions of the first type for a special range of parameter values and find they are unstable.

I. BIFURCATION OF SINGULAR SOLUTIONS IN REVERSIBLE SYSTEMS

1. Formulation of the problem

We consider a differential equation

$$\frac{du}{dx} = u' = A(\mu)u + B(\mu,u), \tag{1.1}$$

where μ is a real parameter and u is in a Banach space Y. We assume:

(i) $A(\mu)$ is of the form $A(\mu) = A_0 + A_1(\mu)$, where $A_0 = A(0)$ is a closed, densely defined linear operator in Y and $A_1(\mu) \in \mathcal{L}(Y)$ is a C^∞- function of μ.

(ii) $B : R \times Y \to Y$ is of class C^∞ and $\|B(\mu,u)\| = 0(\|u\|^2)$ as $u \to 0$.

(iii) Equation (1.1) is reversible in the sense of Moser [26], i.e. there exists a linear isometry $R \in \mathcal{L}(Y)$ such that

$R^2 = id$, $A(\mu)R = -RA(\mu)$ and $B(\mu,Ru) = -RB(\mu,u)$.

(iv) A_0 has an isolated algebraically two-fold but geometrically simple eigenvalue 0.

Let N denote the generalized nullspace of A_0, and M the complementary subspace of Y which is invariant under A_0. It easily follows from (iii) that M and N are invariant under R. Moreover, it is not difficult to prove that $R|_N$ has the simple eigenvalues $+1$ and -1.

(v) M has a decomposition $M = M^+ + M^-$, where M^+ and M^- are invariant under A_0, $M^- = RM^+$ Moreover, $-A_0|_{M^+}$ generates a strongly continuous semigroup of negative type, i.e., for $x > 0$ we have $\|e^{-A_o x}\| < Ce^{-\gamma x}$ with positive constants C and γ. It is a simple consequence that on M^- we have $\|e^{+A_o x}\| < Ce^{-\gamma x}$.

We write $u = (v,w,z)$, where v and w denote the components in N and $z \in M$. Without restricting generality we may assume that R takes (v,w) to $(v,-w)$. Equation (1.1) is then rewritten as follows:

$$(1.2)\quad \begin{aligned} V' &= \alpha(\mu)w + \gamma(\mu)vw + \sigma(\mu)w^3 + wb^*(\mu)z + 0(|v|^2|w| + \\ &\quad |w|^3|v| + \|z\|(|\mu| + |v| + \|z\| + |w|^2)) \\ w' &= \beta(\mu)v + \delta(\mu)v^2 + \zeta(\mu)w^2 + 0(|v|^3 + |w|^2|v| + \|z\| \cdot \\ &\quad (|v| + |w| + \|z\| + |\mu|)), \\ z' &= \tilde{A}(\mu)z + w^2 a(\mu) + 0(\|z\|(|v| + |w| + \|z\|) + |v|^2 + |v|\ |w| \\ &\quad + |w|^3 + |\mu|(|v| + |w|)), \end{aligned}$$

where $\alpha(\mu)$, $\beta(\mu)$, $\delta(\mu)$, $\zeta(\mu)$, and $\sigma(\mu)$ are real numbers, $a(\mu) \in M$, $b^*(\mu) \in M^*$ (the dual of M). and $\tilde{A}(\mu)$ is a linear operator in M. We shall distinguish the following generic cases:

Case 1:

$\beta_0 = \beta(0) = 0$, $\beta_1 = \frac{d}{d\mu}\beta(\mu)|_{\mu=0} \neq 0$, $\delta_0 = \delta(0) \neq 0$.

Then we put $\mu = \pm\varepsilon^2$ and introduce the scaling

$v \to \varepsilon^2 v$, $w \to \varepsilon^3 w$, $z \to \varepsilon^3 z$, $x \to \frac{x}{\varepsilon}$. We obtain:

$$(1.3)\quad \begin{aligned} v' &= \alpha_0 w + 0(|\varepsilon|) \\ w' &= \pm\beta_1 v + \delta_0 v^2 + 0(|\varepsilon|) \\ \varepsilon z' &= \tilde{A}(0)z + 0(|\varepsilon|) \end{aligned}$$

Case 2:

$\alpha_0 = \alpha(0) = 0$, $\alpha_1 = \frac{d}{d\mu}\alpha(\mu)|_{\mu=0} \neq 0$, $\beta(0) \neq 0$, $\beta(0)\tilde{\sigma}(0) - \gamma(0)\zeta(\emptyset) = 0$,

where $\tilde{\sigma}(0) = \sigma(0) + b^*(0)\tilde{A}(0)^{-1}a(0)$.

We then put $\tilde{z} = z - w^2\tilde{A}(0)^{-1}a(0)$ and introduce the scaling

$v \to \varepsilon^2 v$, $w \to \varepsilon w$, $\tilde{z} \to \varepsilon^2\tilde{z}$, $x \to \frac{x}{\varepsilon}$. We obtain

$$(1.4)\quad \begin{aligned} v' &= \pm\alpha_1 w + \gamma_0 vw + \tilde{\sigma}_0 w^3 + 0(\|\tilde{z}\| + |\varepsilon|), \\ w' &= \beta_0 v + \zeta_0 w^2 + 0(|\varepsilon|), \\ \varepsilon\tilde{z}' &= \tilde{A}(0)\tilde{z} + 0(|\varepsilon|). \end{aligned}$$

2. Singular solutions for $\varepsilon = 0$

We are now going to discuss the existence of trajectories connecting saddle points for $\varepsilon = 0$. Since $\tilde{A}(0)$ is nonsingular, $\varepsilon = 0$ immediately yields $z = 0$ both for (1.3) and (1.4), and we are left with a two dimensional problem in either case. We start with the easier case 1.

Case 1:

We assume $\alpha_0 > 0$, $\beta_1 < 0$, which can easily be achieved by replacing μ and v by $-\mu$ and/or $-v$ if necessary. For $\varepsilon = 0$ (1.3) reads

$$\begin{aligned} v' &= \alpha_0 w, \\ w' &= \pm\beta_1 v + \delta_0 v^2, \end{aligned} \tag{2.1}$$

which is a Hamiltonian system, i.e. the Hamiltonian

$$H = w^2 \mp \frac{\beta_1}{\alpha_0} v^2 - \frac{2\delta_0}{3\alpha_0} v^3$$

is constant along trajectories. This implies the following

Proposition 2.1:

If the plus sign is chosen in (2.1) (corresponding to $\mu > 0$), then the fixed point $v = -\beta_1/\delta_0$, $w = 0$ is a saddle point, which is connected to itself by a separatix, if the minus sign is chosen, the same holds for the fixed point 0.

Case 2:

(1.4) reads for $\varepsilon = 0$:

$$\begin{aligned} v' &= \pm\alpha_1 w + \gamma_0 vw + \tilde{\sigma}_0 w^3 , \\ w' &= \beta_0 v + \zeta_0 w^2 . \end{aligned} \tag{2.2}$$

Again we may assume $\beta_0 > 0$, $\alpha_1 < 0$. Stationary solutions of (2.2) are given by

$$v = w = 0 \quad \text{or} \quad v = \pm\alpha_1\zeta_0/(\beta_0\tilde{\sigma}_0 - \gamma_0\zeta_0) , \; w^2 = \mp \alpha_1\beta_0/(\beta_0\tilde{\sigma}_0 - \gamma_0\zeta_0) .$$

Hence nontrivial fixed points exist for $\mu > 0$ (i.e. for the choice of the $+$ sign in (2.2)) if

$$\beta_0 \tilde{\sigma}_0 - \gamma_0 \zeta_0 > 0. \tag{2.3}$$

A simple calculation shows that in this case the nontrivial fixed points are saddle points. Separatrices connecting the two nontrivial fixed points are found as follows:

We look for invariant parabolae of the form $v = aw^2 + b$. This curve is invariant under the flow of the differential equation (2.2) iff: $v' = 2aww'$. If (2.2) is inserted into this equation, a short calculation shows that there are in fact two invariant parabolae for (2.2), namely, we get

$$a = \frac{\frac{1}{2}\gamma_0 - \zeta_0 \pm \sqrt{(\frac{1}{2}\gamma_0 - \zeta_0)^2 + 2\tilde{\sigma}_0 \beta_0}}{2\beta_0},$$

$$b = \frac{\pm \alpha_1}{-\frac{1}{2}\gamma_0 - \zeta_0 \pm \sqrt{(\frac{1}{2}\gamma_0 - \zeta_0)^2 + 2\tilde{\sigma}_0 \beta_0}}.$$

It can be checked that the nontrivial fixed points are in fact on these parabolae. We leave the calculations to the reader. That the two parabolae are the only trajectories connecting the two saddle points follows from the uniqueness of stable and unstable manifolds, for which we refer to [30]. Altogether we find

Proposition 2.2:

If (2.3) is satisfied and the $+$ sign is chosen in (2.2) (which corresponds to $\mu > 0$), then there exist two saddle points, which are connected to each other by two trajectories. The solutions of (2.1) and (2.2) are illustrated by the following diagrams:

(2.1), + sign

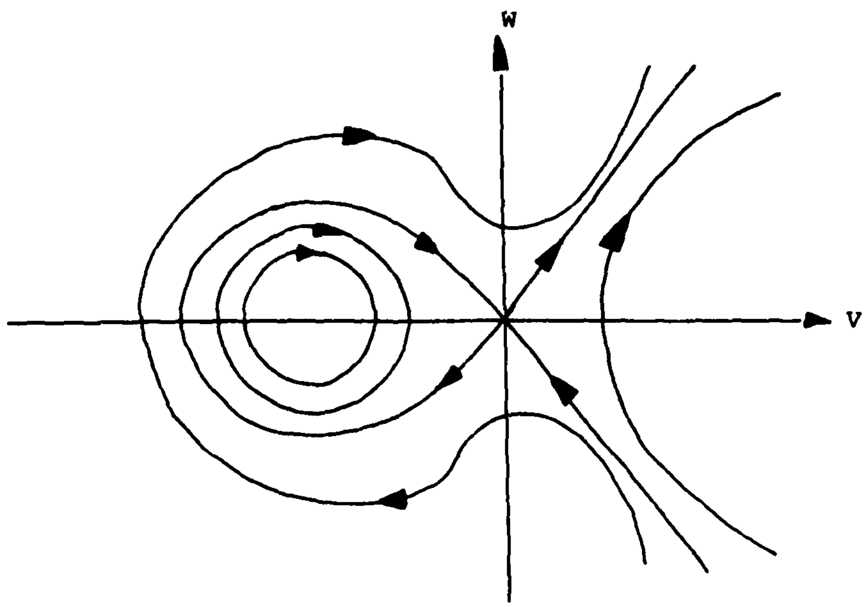


Fig. 2.1

(2.2), - sign

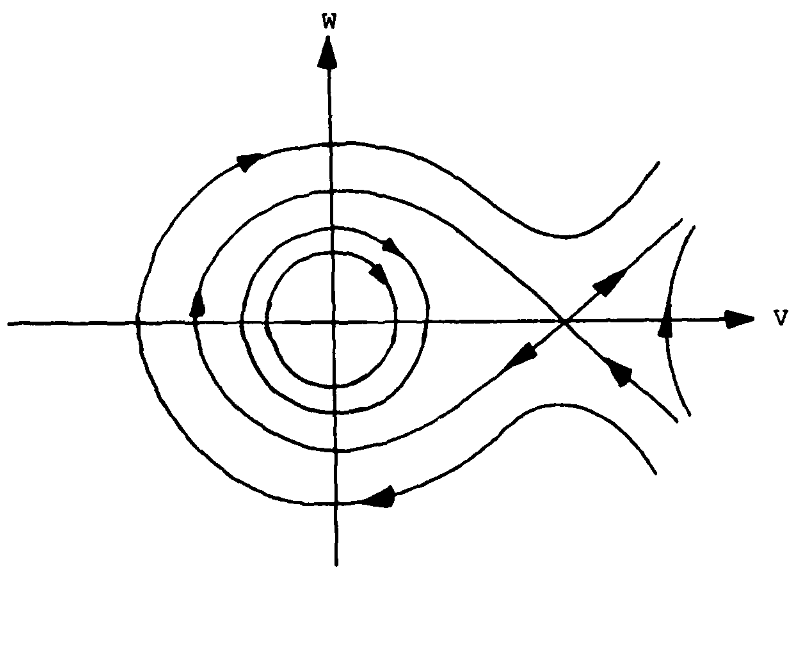


Fig. 2.2

(2.2), + sign, (2.3) holds

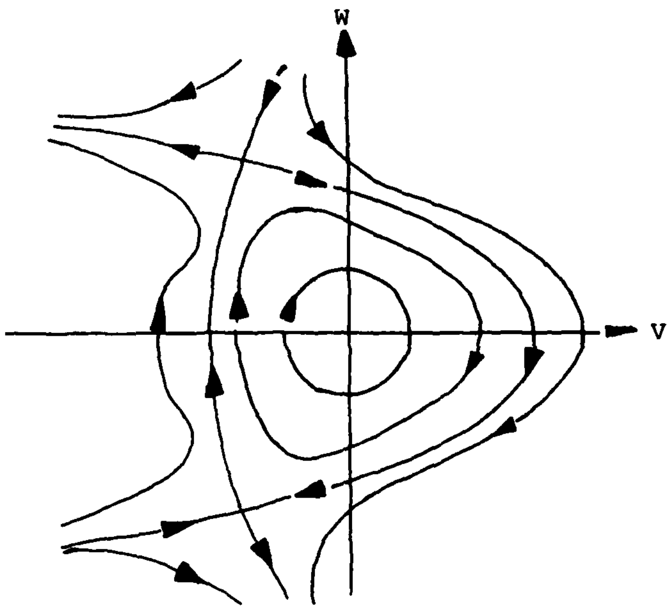


Fig. 2.3

3. The linearization at the singular solutions

All the trajectories connecting saddle points which we have found in § 2 are symmetric with respect to the w-axis. This means that among the one-parameter family of solutions represented by such a trajectory there is one solution $y_0(x) = (v_0(x), w_0(x))$ satisfying $Ry_0(x) = y_0(-x)$. We denote the linearization of (2.1) resp. (2.2) at $y_0(x)$ by

$$y' = C_0(x)y$$

Using the reversibility and the fact that $Ry_0(x) = y_0(-x)$ one finds: $C_0(-x)R = -RC_0(x)$. We shall prove

Theorem 3.1

For each $f(x) = (f_1(x), f_2(x)) \in F_m = \{f \in C_b^m(R,R^2) \mid Rf(x) = -f(x),$ $\lim_{x\to\infty} f^{(k)}(x)$, $\lim_{x\to-\infty} f^{(k)}(x)$ exist for $0\le k\le m\}$ there exists one and only one $y(x) = (v(x), w(x)) \in U_{m+1} = \{y \in C_b^{m+1}(R,R^2) \mid Ry(x) = y(-x)$, $\lim_{x\to\infty} y^{(k)}(x)$, $\lim_{x\to-\infty} y^{(k)}(x)$ exist for $0\le k\le m+1\}$ solving the inhomogeneous equation

$$y' - C_0(x)y = f \tag{3.1}$$

Here $C_b^m(R,R^2)$ denotes the Banach space of all functions $R \to R^2$ having m continuous bounded derivatives.

Proof:

For $x \to \pm\infty$, $y_0(x)$ converges to a saddle point, and one easily concludes from the stable manifold theorem [30] that the convergence is exponential. Repeated differentiation of (2.1) resp. (2.2) then yields the result that all derivatives of y_0 converge to 0 exponentially. This implies the following properties of $C_0(x)$: For $x \to \pm\infty$, $C_0(x)$ converges to the linearization at a saddle point, i.e. $\lim_{x\to\infty} C_0(x)$ exists, and this matrix has one positive and one negative eigenvalue. Moreover $d^m/dx^m(C_0(x))$ converges to 0 exponentially for each $m > 0$. From this we see that it is sufficient to prove the theorem for $m = 0$, the rest following from (3.1) by repeated differentiation. We rewrite (3.1) in the form

$$y' - \hat{C}y - \tilde{C}_0(x)y = f ,$$

where $\hat{C}$ is constant, and $\lim_{x\to\infty} \tilde{C}_0(x) = 0$. From the fact that $\hat{C}$ has one positive and one negative eigenvalue we conclude that for each $f \in C_b([X,\infty),R^2)$ the equation

$$y' - \hat{C}y = f \tag{3.2}$$

has a solution $y \in C_b^1([X,\infty),R^2)$, which is determined up to one arbitrary initial condition at $x = X$. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} y(x)$ exists as well. Let P denote a projection of R^2 onto a one-dimensional subspace such that the prescription of $Py(X)$ determines the solution of (3.2) uniquely. Then the mapping $y \to (y' - \hat{C}y, Py(X))$ is an isomorphism from $U_X = \{y \in C_b^1([X,\infty),R^2) | \lim_{x\to\infty} y(x), \lim_{x\to\infty} y'(x) \text{ exist}\}$ onto $(F_X = \{f \in C_b([X,\infty),R^2) | \lim_{x\to\infty} f(x) \text{ exists }\}) \times R$. Since (3.2) is autonomous, the norm of this isomorphism and its inverse are independent of X. On the other hand, the norm of the mapping $y \to \tilde{C}_0(x)y$ from U_X onto F_X tends to 0 as $X \to \infty$, i.e. for sufficiently large X the mapping $y \to (y' - \hat{C}y_0 - \tilde{C}(x)y, Py(X))$ is still an isomorphism from U_X onto $F_X \times R$. That means, given any $f \in F_0$, there exists a bounded solution y to (3.1) on the interval $[X,\infty)$, which is determined up to one initial condition at $x = X$. Since the initial value problem for (3.1) is uniquely solvable, this solution on $[X,\infty)$ extends to a solution on $[0,\infty)$, and we have one free initial condition at $x = 0$. This remaining free initial condition is matched by adding a multiple of the solution $y_0'(x)$ of the homogeneous problem. The requirement $y(x) = Ry(-x)$ implies $w(0) = 0$, which is achieved by one and only one choice of the initial condition, since $w_0'(0) \neq 0$. This determines the "half-sided" solution on $[0,\infty)$ uniquely. Since $Rf(x) = -f(-x)$, we have further: If $y(x)$ solves (3.1) on $[0,\infty)$, then $Ry(-x)$ solves (3.1) on

$(-\infty,0]$ (use the fact that $C_0(-x)R=-RC_0(x)$). Hence the half-sided solution extends to a solution on all of R, and the theorem is proved.

4. Existence of singular solutions for $\varepsilon \neq 0$

The goal of this paragraph is to prove that there exists a branch of singular solutions in a neighborhood of $\varepsilon = 0$. For this let $y_0(x)$ be as in § 3 and put $h(x) = (v(x),w(x)) - y_0(x)$. (1.3) or (1.4) then takes the form

$$
\begin{aligned}
h' &= C_0(x)h + f(\varepsilon,h,z,x) , \\
\varepsilon z' &= \tilde{A}(0)z + g(\varepsilon,h,z,x) .
\end{aligned}
\tag{4.1}
$$

Here $C_0(x)$ is as in § 3 and we have $\|f(\varepsilon,h,z,x)\| = O(|\varepsilon| + \|z\| + \|h\|^2)$, $\|g(\varepsilon,h,z,x)\| = O(|\varepsilon|)$. For each $m \in N$ the mapping $(\varepsilon,h,z) \to (f(\varepsilon,h,z,x), g(\varepsilon,h,z,x))$ is a C^∞-mapping from $R \times U_m(Y)$ to $F_m(Y)$, where we denote

$$U_m(Y) = \{u \in C_b^m(R,Y) \mid \lim_{x\to\infty} u^{(k)}(x) , \lim_{x\to-\infty} u^{(k)}(x) \text{ exist for } 0 \leq k \leq m, \; Ru(x) = u(-x)\} \text{ and}$$

$$F_m(Y) = \{f \in C_b^m(R,Y) \mid \lim_{x\to\infty} f^{(k)}(x), \lim_{x\to-\infty} f^{(k)}(x) \text{ exist for } 0 \leq k \leq m, \; Rf(x) = -f(x)\}$$

We write (4.1) as follows:

$$
\begin{aligned}
h - \left(\frac{d}{dx} - C_0(x)\right)^{-1} f(\varepsilon,h,z,x) &= 0 \\
z - \left(\varepsilon \frac{d}{dx} - \tilde{A}(0)\right)^{-1} g(\varepsilon,h,z,x) &= 0
\end{aligned}
\tag{4.2}
$$

We wish to apply an implicit function argument to establish the existence of solutions in $U_m(Y)$ for $\varepsilon \neq 0$. A problem is caused by the fact that $\varepsilon \frac{d}{dx}$ is a relatively unbounded perturbation to $\tilde{A}(0)$. It is, however, not difficult to show the following (the details are in [33]):

Lemma 4.1:

(i) The mapping $(\varepsilon,z) \mapsto (\varepsilon \frac{d}{dx} - \tilde{A}(0))^{-1}z$ from $R \times F_m(M)$ into $U_m(M)$ is continuous near $\varepsilon = 0$ (M as defined in (iv) of § 1)

(ii) The operator norm of $(\varepsilon \frac{d}{dx} - \tilde{A}(0))^{-1} : F_m(M) \to U_m(M)$ is uniformly bounded in a neighborhood of $\varepsilon = 0$.

(iii) The mapping $(\varepsilon,z) \to (\varepsilon \frac{d}{dx} - \check{A}(0))^{-1}z$ from $\mathbb{R} \times F_m(M)$ into $U_{m-k}(M)$ is of class C^k.

These properties permit the use of the following abstract theorems which we proved in [32], [33].

<u>Theorem 4.2</u>

Let X,Y and Z be Banach spaces, U a neighborhood of $(0,0)$ in $X \times Y$, and $F: U \to Z$ a mapping having the following properties:

(i) $F(0,0) = 0$.

(ii) F is continuous.

(iii) F is continuously differentiable with respect to y for each fixed x.

(iv) $D_yF(0,0): Y \to Z$ is an isomorphism.

(v) D_yF is continuous at the point $(0,0)$.

Then the equation $F(x,y) = 0$ has a unique continuous resolution $y = f(x)$ in some neighborhood of $(0,0)$.

<u>Theorem 4.3</u>

Let $Y^{(k)}$ resp. $Z^{(k)} (k = 0,1,\dots N)$ be two hierarchies of Banach spaces such that $Y^{(k)} \subset Y^{(k+1)}$, $Z^{(k)} \subset Z^{(k+1)}$, the imbeddings being continuous. Let X be a finite dimensional Banach space and F a mapping from a neighborhood U of 0 in $X \times Y^{(N)}$ into $Z^{(N)}$ having the following properties:

(i) $F(U \cap (X \times Y^{(k)})) \subset Z^{(k)}$ $\quad k = 0,1,\dots N$

(ii) For each fixed k, $F_k := F|_{U \cap (X \times Y^{(k)})}$ satisfies the conditions of theorem 4.2, when it is considered as a mapping from $X \times Y^{(k)}$ into $Z^{(k)}$. For x fixed, $F_k(x,.)$ is a smooth (i.e. sufficiently often differentiable) mapping.

(iii) $F: X \times Y^{(k)} \to Z^{(k+m)}$ is of class C^m for each $k = 0, 1,\dots, N$

and $m \leq N - k$.

(iv) The mapping $(x,y,u^1,\ldots,u^j) \to z = D_x^i D_y^j F(x,y)(u^1,\ldots,u^j)$ is a continuous mapping from $X \times Y^{(k)} \times (Y^{(k)})^j$ into $\mathcal{L}^i(X,Y^{(k+i)})$.

Then the following holds:

The solution $y = f(x) \in Y^{(0)}$ existing by theorem 4.2 is a C^m-function of x in some neighborhood V_m of 0, if y is regarded as an element of $Y^{(m)}$.

Identifying X with $\mathbb{R}$, $Y^{(k)}$ and $Z^{(k)}$ with $U_{m-k}(Y)$, we get from these theorems.

<u>*Theorem 4.4:*</u>

For each $m \in \mathbb{N}$ there exists a neighborhood $V^{(m)}$ of $(0,0) \in \mathbb{R} \times U_m(Y)$ such that in $V^{(m)}$ equation (4.2) has a unique resolution $h = h(\varepsilon)$, $z = z(\varepsilon)$. If this solution is considered lying in $U_{m-k}(Y)$, then in some neighborhood of $\varepsilon = 0$, it is a C^k-function of ε.

<u>Remarks</u>

1. In order to carry out the iteration procedure in the proof of the implicit function theorem, equations of type (3.1) have to be solved. Although the explicit solution of (3.1) has not been employed in the proofs, it can easily be obtained modulo integrations, since one integral of the homogeneous equation is known.
2. In the case 1 of § 1,2 there are singular solutions approaching 0 for $x \to \pm\infty$. Using the stable manifold theorem it can be shown that the convergence is exponential, and the singular solutions are therefore in L^p for each $p \geq 1$. But do they bifurcate in L^p? To answer this question, we must see how the scaling introduced in § 1 affects the L^p-norm. In the generic case considered here u has been scaled by ε^2 and x by ε^{-1}, which gives a factor of $\varepsilon^{2-1/p}$ in the L^p-norm.

Therefore we have a bifurcation in the space L^p for each $p > 1$. This need no longer be true if degeneracies occur and different scaling factors must be used. Küpper and Riemer [24] have considered the example

$$-u'' - |u|^r f(u) = \lambda u$$

where $f(0) < 0$ and $r < 1$.

Our method described above applies to this example with the scaling $\lambda = -\varepsilon^2$, $u \to \varepsilon^{2/(r-1)}u$, $x \to \varepsilon^{-1}x$. In the L^p-norm this gives a factor of $\varepsilon^{2/(r-1)-1/p}$. This exponent is greater than zero if $r < 2p + 1$. This agrees with the result obtained in [24] for the case $p = 2$.

5. Global existence of singular solutions

In the preceding chapters we have proved the existence of branches of singular solutions in the neighborhood of some bifurcation point. This chapter deals with the problem, how far these branches can be continued. The main tool of the analysis will be the theory of degrees of mappings. Under appropriate conditions, a degree can be associated with our singular solutions in a quite similar way as with solutions representing travelling waves [39].

Since the definition of a degree requires a compactness assumption, we impose the following condition in addition to (i) - (v) of § 1:

(vi) $-A_0|_{M^+}$ and $A_0|_{M^-}$, respectively, generate analytic semigroups and have compact resolvents.

As a consequence, fractional powers $(-A_0)^\alpha$ and A_0^α are defined as operators in M^- and M^+, respectively. We say briefly that $y \in Y$ is on $D(A_0^\alpha)$, if its M^--component is in $D(-A_0^\alpha)$ and its M^+-component is in $D(A_0^\alpha)$. It follows from the compactness of the resolvent that, for any $0 < \alpha < 1$, $D(A_0^\alpha)$ is compactly embedded in Y.

The result we want to show is that branches of solutions can be continued unless some norm of the solution approaches infinity. Obviously, this can only be expected, if the nonlinear terms in the equation remain bounded on bounded sets. We therefore assume

(vii) The nonlinear operator B maps bounded sets into bounded sets.

In our exposition we focus on the case 1 of §1. Again we assume $\alpha_0 > 0$, $\beta_1 < 0$. In this case we have proved that a branch of trajectories connecting the saddle point 0 to itself exists for $\mu < 0$.

We shall be concerned with solutions lying in the following spaces.

Definition 5.1:

Let Y be the same Banach space as in § 1. Then

$X^n_\sigma(Y) := \{y(x) \in C^n(\mathbb{R},Y) | Ry(x) = y(-x),\ \sup_{x\in\mathbb{R}} e^{\sigma|x|}\|y^{(k)}(x)\| < \infty \text{ for } k = 0,1,\ldots n\}$

$y^{(k)}$ denotes the kth derivative of y.

In §§ 1-4 we have shown that, for ε small, (1.3) has two solutions in $X^n_\sigma(Y)$, namely 0 and the singular solution. Both are isolated, and the linearization of (1.3) at either solution is nondegenerate.

For convenience, we let y denote (v,w,z), $L(\varepsilon)$ the linearization of the right hand side at 0, $N(\varepsilon,y)$ the nonlinear part of the right hand side of (1.3), and φ_ε the mapping $(v,w,\varepsilon z)$. The (1.3) can be rewritten in the form

$$y = \left(\frac{d}{dx} - \varphi_\varepsilon^{-1}L(\varepsilon)\right)^{-1} \varphi_\varepsilon^{-1}N(\varepsilon,y) \tag{5.1}$$

Let $\Gamma = [\varepsilon_1,\varepsilon_2]$ denote an interval with the following properties:

1. $\varepsilon_2 > \varepsilon_1 > 0$, ε_1 small
2. For each $\varepsilon \in \Gamma$, the solution 0 of equation (1.3) is a saddle point, and the real parts of the eigenvalues of $\varphi_\varepsilon^{-1}L(\varepsilon)$ have a positive distance δ (uniformly in ε) from the strip

$$\{\lambda \in \mathbb{C} | -\sigma < \text{Re}\ \lambda < \sigma\}$$

The crucial property for a global bifurcation result is the following

<u>Propositon 5.1</u>

Let D be any ball in $X^n_\sigma(Y)$. Then the right hand side of equation (5.1) represents a completely continuous mapping from $\Gamma \times D$ into $X^n_\sigma(Y)$

Proof:

It follows from assumption (vii) that N is continuous and bounded into $X^n_{2\sigma}(Y)$. Moreover, our assumptions on A_0 and Γ imply that $\left(\frac{d}{dx} - \varphi_\varepsilon^{-1}L(\varepsilon)\right)^{-1}$ is a bounded continuous mapping from $\Gamma \times X^n_{2\sigma}(Y)$ into $X^{n+\alpha}_\zeta(D(A_0^\eta))$ for some $\alpha,\eta > 0$ and $\zeta < \min(2\sigma,\sigma+\delta)$. The rest follows from the compactness of the embedding $D(A_0^\eta) \to Y$ and the Arzela-

Ascoli theorem.

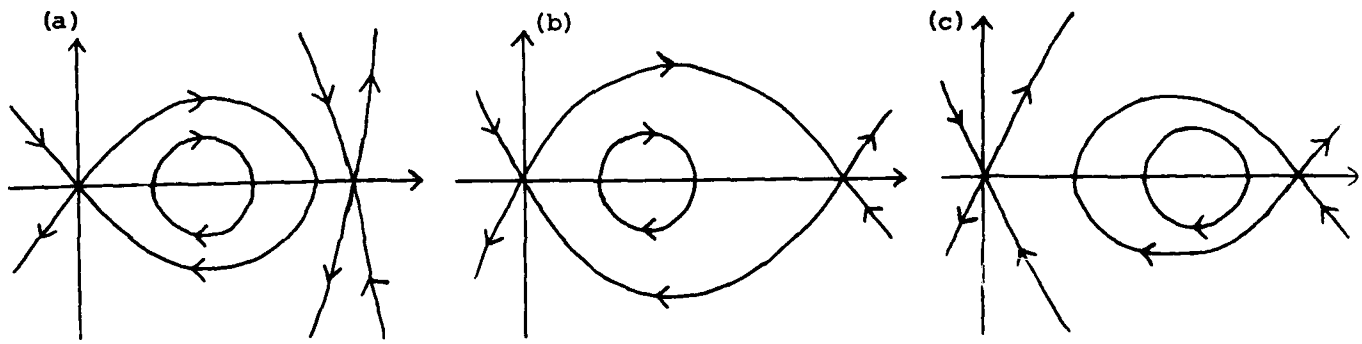
Degree theory now gives the following result (cf. [37]):

Theorem 5.2:

The branch of singular solutions to (1.3) provided by theorem 4.4 must either leave Γ or the norms of the solutions are unbounded in $X^n_\sigma(Y)$.

Remarks:

1. The reader is cautioned that unboundedness in $X^n_\sigma(Y)$ does not mean unboundedness in the sup-norm. In fact, branches of singular solutions can terminate by a mechanism like the one indicated in the following sequence of diagrams.



For any $\sigma > 0$, the norm in X^n_σ tends to ∞ when passing from situation (a) to situation (b), and the branch of solutions connecting 0 to itself terminates its existence. Similar phenomena must be expected, when singular solutions come close to an invariant set other than a stationary point.

2. The above theorem does not exclude the possibility that a branch goes back to $\varepsilon = 0$. However, branches can not return to the point they bifurcated from, as one can see from a local uniqueness result. Namely, there exists a center manifold $z = g(v,w,\mu)$ for equation (1.1), and for μ near 0 all uniformly small solutions must lie on the center manifold. This follows from the following argument:

If we put $\hat{z} = z - g(v,w,\mu)$, we obtain for $\hat{z}$ a differential equation having the following form

$$\hat{z}' = \tilde{A}(0)\hat{z} + O((|\mu| + |v| + |w|)) \|\hat{z}\|) + O(\|\hat{z}\|^2)$$

For small $|v|$, $|w|$, $|\mu|$, the only small solution of this is $\hat{z} = 0$, as follows from the implicit function theorem. From this and elementary considerations about two-dimensional flows it follows that near $\mu = 0$ the saddle point 0 and the singular solution provided by theorem 4.4 are the only small solutions in X_σ^n.

For the case 2 of § 1, analogous considerations are possible, but now $L(\varepsilon)$ should be replaced by an appropriate operator $L(\varepsilon,x)$ which converges to the linearization at the limiting fixed points for $x \to \pm\infty$.

II. APPLICATIONS TO REACTION-DIFFUSION MODELS

6. Oscillating singular solutions connected with Hopf bifurcations

We consider a general chemical reaction model given by an equation

$$\frac{\partial u}{\partial t} = F(\mu,u) + D\frac{\partial^2 u}{\partial x^2} \tag{6.1}$$

where $u \in \mathbb{R}^n$, $\mu \in \mathbb{R}$. F is a smooth nonlinear function. D is a diagonal matrix, which is strictly positive definite. We assume

(i) For μ in some neighborhood of 0, there exists a solution $u = u_0(\mu) \in \mathbb{R}^n$ to the equation $F(\mu,u) = 0$, and u_0 is a C^∞-function of μ.

(ii) The matrix $D_uF(0,u_0(0))$ has the algebraically simple imaginary eigenvalues $\pm i\omega_0$, the rest of the spectrum lies in the left half plane.

(iii) Let $\lambda(\mu)$ denote the branch of eigenvalues of $D_uF(\mu,u_0(\mu))$, which goes through $i\omega_0$ at $\mu = 0$. Then $\frac{d}{d\mu}\operatorname{Re}\lambda(\mu)|_{\mu=0} \neq 0$.

(iv) For each $\gamma > 0$, the spectrum of $D_uF(0,u_0(0)) - D\gamma$ lies in the left half plane.

It is well known [14] that conditions (i)-(iii) imply the existence of a branch of x-independent time-periodic solutions emerging from the bifurcation point $u = u_0(0)$, $\mu = 0$. The conditions (i)-(iv) have been verified in quite a few reaction diffusion models, e.g. the "Brusselator" [1], [12] and the Field-Noyes model of the B-Z reaction [6], [13], [17], [18], [27], [31].

In this chapter we shall study time-periodic space-dependent solutions of equations (6.1). We rewrite this equation in the form

$$\frac{1}{\omega}u'' = D^{-1}\left(\frac{\partial u}{\partial t} - \frac{1}{\omega}F(\mu,u)\right) \tag{6.2}$$

The factor $\omega > 0$ has been introduced in order to normalize the period to

2π. After appropriate scaling of x we may drop $\frac{1}{\omega}$ on the left side. Moreover, we write (6.2) as a first order system.

$$u' = v$$
(6.3)
$$v' = D^{-1}\left(\frac{\partial u}{\partial t} - \frac{1}{\omega} F(\mu,u)\right)$$

Let now $\ell^1_\alpha(\mathbb{R}^n)$ denote the space of all 2π-periodic functions $y\colon \mathbb{R} \to \mathbb{R}^n$, for which $\|y\|_\alpha := \sum_{k\in\mathbb{Z}} (|k|^\alpha + 1)\,\|y^{(k)}\| < \infty$, where the $y^{(k)}$ are the Fourier coefficients: $y(t) = \sum_{k\in\mathbb{Z}} y^{(k)}e^{ikt}$. We shall seek solutions to (6.3) in the space $Y_m = \{(u,v)\,|\,u \in \ell^1_m(\mathbb{R}^n),\ v \in \ell^1_{m-1/2}(\mathbb{R}^n)\}$, where m is an arbitrary positive integer. Clearly, the mapping $(\mu,\omega,(u,v)) \mapsto (0,-D^{-1}\omega^{-1}F(\mu,u))$ is smooth from $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times Y_m$ into Y_m, moreover, the operator $(u,v) \mapsto (v, D^{-1}\frac{\partial u}{\partial t})$ is densely defined in Y_m and closed. Equation (6.3) is reversible under both mappings $R : (u,v) \mapsto (u,-v)$ and $\hat{R} : (u(t),v(t)) \mapsto (u(t+\pi),-v(t+\pi))$.

We now discuss the spectrum of the linearization of the right side of (6.3) at the point $\omega = \omega_0$, $\mu = 0$, $u = u_0(0)$. The relevant properties of this spectrum are described by the following lemma.

<u>Lemma 6.1:</u>

Assume (i)-(iv) and (v) stated below hold. Then the operator $A : (u,v) \mapsto (v, D^{-1}\frac{\partial u}{\partial t} - (D\omega_0)^{-1} D_uF(0,u_0(\mu))u)$ has the isolated algebraically four-fold and geometrically two-fold eigenvalue 0. Let N denote the generalized nullspace and M a complementary invariant subspace. Then $A|_M$ satisfies condition (v) of § 1 (even (vi) of § 5, but we shall not use this.)

Proof:

We first note that A acts Fourier-componentwise, i.e. we have

$$A(\sum(u^{(k)},v^{(k)})e^{ikt} = \sum A^{(k)}(u^{(k)},v^{(k)})e^{ikt}$$

where

$$A^{(k)} = \begin{pmatrix} 0 & 1 \\ D^{-1}ik - (D\omega_0)^{-1}D_uF(0,u_0(0)) & 0 \end{pmatrix}$$

Thus the eigenvalues of $A^{(k)}$ are the square roots of the eigenvalues of the matrix

$$D^{-1}ik - (D\omega_0)^{-1}D_uF(0,u_0(0))$$

and $A^{(k)}$ has an imaginary eigenvalue iff this matrix has a negative real eigenvalue. Let now be

$$(D^{-1}ik - (D\omega_0)^{-1}D_uF(0,u_0(0)))y = -\gamma y \qquad \gamma > 0$$

This yields

$$D_uF(0,u_0(0))y - \omega_0\gamma Dy = ik\omega_0 y$$

For $\gamma \neq 0$ this is impossible by condition (iv), and $\gamma = 0$ yields $k = \pm 1$, since $\pm i\omega_0$ are the only imaginary eigenvalues of $D_uF(0,u_0(0))$. We see therefore that $A^{(k)}$ has no imaginary eigenvalues for $k \neq \pm 1$, and the only imaginary eigenvalue for $k = \pm 1$ is equal to 0. To prove the statement concerning the multiplicity, it must be shown that $\lambda = 0$ is an algebraically simple eigenvalue of

$$D^{-1}i - (D\omega_0)^{-1}D_uF(0,u_0(0))$$

It is easy to see that the nullspace of this matrix is spanned by the eigenvector of $D_uF(0,u_0(0))$ to the eigenvalue $i\omega_0$, whence the eigenvalue $\lambda = 0$ is geometrically simple. Assume now that

$$D_uF(0,u_0(0))y = i\omega_0 y$$

and

$$y = (D^{-1}i - (D\omega_0)^{-1}D_uF(0,u_0(0))z$$

This yields

$$\omega_0 Dy = (i\omega_0 - D_uF(0,u_o(0)))z$$

We shall therefore assume

(v) With y denoting the eigenvector of $D_uF(0,u_0(0))$ corresponding to the eigenvalue $i\omega_0$, Dy is not in the range of $i\omega_0 - D_uF(0,u_0(0))$.

Clearly (v) is a generic condition, which guarantees that the eigenvalue is algebraically simple (in particular, (v) follows from (ii) if D is the identity matrix).

We must now verify that the spectrum of A is actually given by the eigenvalues of the $A^{(k)}$ and that (v) of § 1 holds. This will be a consequence of the following:

There exists an isomorphism T of Y_m acting Fourier componentwise:

$$T \sum_{k\in Z} (u^{(k)},v^{(k)})e^{ikt} = \sum_{k\in Z} T^{(k)}(u^{(k)},v^{(k)})e^{ikt}$$

such that for large $|k|$, let us say for $|k| > k_0$ the matrix $(T^{(k)})^{-1}A^{(k)}T^{(k)}$ consists of a diagonal part and a rest term which has a norm of the order of magnitude $|k|^{-1/2}$. Namely, put for $|k| > k_0$

$$T^{(k)} = \begin{pmatrix} 1 & \sqrt{\frac{D}{ik}} \\ -\sqrt{\frac{ik}{D}} & 1 \end{pmatrix}$$

This yields

$$(T^{(k)})^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{D}{ik}} \\ \frac{1}{2}\sqrt{\frac{ik}{D}} & \frac{1}{2} \end{pmatrix}$$

$$(T^{(k)})^{-1}A^{(k)}T^{(k)} = \begin{pmatrix} -\sqrt{\frac{ik}{D}} + O(|k|^{-1/2}) & O(|k|^{-1}) \\ O(1) & \sqrt{\frac{ik}{D}} + O(|k|^{-1/2}) \end{pmatrix}$$

i.e. $(T^{(k)})^{-1}A^{(k)}T^{(k)}$ is the sum of diagonal matrix and a remainder term which has an operator norm of the order of magnitude $|k|^{-1/2}$ (recall the definition of the norm in Y_m).

It is now a simple consequence of the Hille-Yosida theorem [25] that property (v) of § 1 holds for the diagonal part, since the necessary resolvent estimates are in this case trivial. A perturbation argument shows that the same holds for the full operator $T^{-1}AT$, and hence for A. This concludes the proof of the lemma.

Again y shall denote the eigenvector of $D_uF(0,u_0(0))$ corresponding to the eigenvalue $i\omega_0$. We decompose (u,v) as follows

$$(u,v) = (u_0(\mu),0) + \alpha_1(y,0)e^{it} + \bar{\alpha}_1(\bar{y},0)e^{-it} + \alpha_2(0,y)e^{it} + \bar{\alpha}_2\ (0,\bar{y})e^{-it} + z$$

where $\alpha_1,\alpha_2 \in C$ and $z \in M$ (as defined under lemma 6.1). Equations (6.3) then assume the form

$$\begin{aligned} \alpha_1' &= \alpha_2 \\ \alpha_2' &= \mu a_1\alpha_1 + a_2\alpha_1^2\bar{\alpha}_1 + C_1(z,\alpha_1) + C_2(z,\bar{\alpha}_1) + (\omega^{-1} - \omega_0^{-1})a_3\alpha_1 + \dots \\ z' &= \check{A}(0)z + \alpha_1^2 d_1 + \alpha_1\bar{\alpha}_1 d_2 + \bar{\alpha}_1^2 d_3 + \dots \end{aligned}$$

Here a_1, a_2, a_3 are complex numbers, $C_{1,2} : M \times C \to C$ are bilinear operators and the d_i are vectors in M. $\check{A}(0)$ denotes $A|_M$. The dots indicate higher order terms.

Analogously as in § 1 we put $z = \check{z} - \check{A}(0)^{-1}\{\alpha_1^2 d_1 + \alpha_1\bar{\alpha}_1 d_2 + \bar{\alpha}_1^2 d_3\}$ and introduce the scaling

$\alpha_1 \to \varepsilon\alpha_1$, $\alpha_2 \to \varepsilon^2\alpha_2$, $\mu = \pm\varepsilon^2, \check{z} \to \varepsilon^2\check{z}$, $\omega^{-1} - \omega_0^{-1} \to \varepsilon^2\hat{\omega}$, $x \to \varepsilon^{-1}x$. We then obtain an equation of the following form

$$
\begin{aligned}
&\alpha_1' = \alpha_2 \\
(6.4) \qquad &\alpha_2' = \pm\, a_1\alpha_1 + \tilde{a}_2\alpha_1^2\bar{\alpha}_1 + a_3\hat{\omega}\alpha_1 + O(|\varepsilon| + \|\tilde{z}\|) \\
&\varepsilon\tilde{z}' = \tilde{A}(0)\tilde{z} + O(|\varepsilon|)
\end{aligned}
$$

For $\varepsilon = 0$ this reduces to

$$
(6.5) \qquad \alpha_1'' = \pm\, a_1\alpha_1 + a_3\hat{\omega}\alpha_1 + \tilde{a}_2\alpha_1^2\bar{\alpha}_1
$$

Apart from a factor of $\nu = \frac{1}{(b^*,Dy)}$ (where b^* denotes the left handed eigenvector of $D_uF(0,u_0(0))$ corresponding to $i\omega_0$, normalized such that $(b^*,y) = 1$), the coefficients a_1, $\tilde{a}_2$ and a_3 are the same which determine the Hopf bifurcation at the order ε^3. It follows from assumption (iii) that $\mathrm{Re}\,\frac{a_1}{\nu}$ is negative. Moreover, a_3 is equal to $i\nu$. $\hat{\omega}$ is an unknown variable, which has to be determined.

Equation (6.5) is the simplest case for a class of equations that have been called "λ-ω -systems" [10], [11], [15], [22], [23]. We now try to solve (6.5) by the ansatz $\alpha_1 = re^{i\varphi}$, $r = C\,\mathrm{sech}\,kx$, $\varphi' = B\tanh kx$. After some elementary calculations this leads to the equations

$$
\begin{aligned}
k^2 - B^2 &= \pm\,\mathrm{Re}\,a_1 + \hat{\omega}\,\mathrm{Re}\,a_3 \\
(6.6) \qquad -2k^2 + B^2 &= C^2\mathrm{Re}\,\tilde{a}_2 \\
-2Bk &= \pm\,\mathrm{Im}\,a_1 + \hat{\omega}\,\mathrm{Im}\,a_3 \\
3Bk &= C^2\,\mathrm{Im}\,\tilde{a}_2
\end{aligned}
$$

From the fourth and second equation of (6.6) we find

$$
3Bk\,\mathrm{Re}\,\tilde{a}_2 + (2k^2 - B^2)\,\mathrm{Im}\,\tilde{a}_2 = 0
$$

which can be solved by $B = \lambda k$, where (provided $\mathrm{Im}\,\hat{a}_2 \neq 0$)

$$\lambda = \frac{-3 \operatorname{Re} \tilde{a}_2 \pm \sqrt{9(\operatorname{Re} \tilde{a}_2)^2 + 8(\operatorname{Im} \tilde{a}_2)^2}}{-2 \operatorname{Im} \tilde{a}_2}$$

According to the fourth equation of (6.6), λ must have the same sign as $\operatorname{Im} \tilde{a}_2$, which is achieved by choosing the minus sign in the numerator.

We now insert $B = \lambda k$ into the first and third equation of (6.6), thus obtaining

$$k^2(1 - \lambda^2) = \pm \operatorname{Re} a_1 + \hat{\omega} \operatorname{Re} a_3$$

$$-2\lambda k^2 = \pm \operatorname{Im} a_1 + \hat{\omega} \operatorname{Im} a_3$$

$$\Longrightarrow k^2 \{(1 - \lambda^2) \operatorname{Im} a_3 + 2\lambda \operatorname{Re} a_3\} = \pm(\operatorname{Re} a_1 \operatorname{Im} a_3 - \operatorname{Im} a_1 \operatorname{Re} a_3)$$

The right side of this last equation is not zero, according to what we have said about a_1 and a_3 above. We can resolve the equations with respect to k and $\hat{\omega}$ if the following holds

(vi) $\qquad (1 - \lambda^2) \operatorname{Im} a_3 + 2\lambda \operatorname{Re} a_3 \neq 0$

If $\operatorname{Im} \tilde{a}_2 = 0$, (6.6) can be solved by $B = 0$, provided that the following holds:

(vii) If $\operatorname{Im} \tilde{a}_2 = 0$, then $\operatorname{Re} \tilde{a}_2$ is negative and $\operatorname{Im} a_3 \neq 0$

We find thus

<u>Proposition 6.2</u>

If (vi) or (vii), respectively, are satisfied, then the ansatz $\alpha_1 = re^{i\varphi}$, $r = C \operatorname{sech} kx$, $\varphi' = B \tanh kx$ leads to a solution $\alpha_1^0(x)$ of (6.6).

Clearly, $\alpha_1^0(x)$ is an even function of x and satisifies $\lim_{x\to\pm\infty} \alpha_1^0(x) = 0$. Moreover, the asymptotic behaviour of $u(x,t)$ for $x \to \infty$ is described by

$$u(x,t) \sim u_0(\mu) + 2C(\varepsilon e^{-k\varepsilon x} e^{i(t+B\varepsilon x)} + \text{c.c.})$$

and for $x \to -\infty$ by

$$u(x,t) \sim u_0(\mu) + 2C(\varepsilon e^{+k\varepsilon x} e^{i(t-B\varepsilon x)} + c.c.)$$

This means that asymptotically we have exponentially damped waves, propogating in opposite directions for $x \to \pm\infty$.

A second ansatz for a solution of (6.6) is $\alpha_1 = re^{i\varphi}$, $r = C \tanh kx$, $\varphi' = B \tanh kx$. This ansatz leads to

$$(6.7) \qquad \begin{aligned} -2k^2 &= \pm \mathrm{Re}\, a_1 + \hat{\omega}\, \mathrm{Re}\, a_3 \\ 3kB &= \pm\, \mathrm{Im}\, a_1 + \hat{\omega}\, \mathrm{Im}\, a_3 \\ 2k^2 - B^2 &= C^2 \mathrm{Re}\, \tilde{a}_2 \\ -3kB &= C^2 \mathrm{Im}\, \tilde{a}_2 \end{aligned}$$

The last two equations can again be solved by the ansatz $B = \lambda k$ with the same expression for λ as before, this time, however, the plus sign must be chosen in front of the square root. The first two equations now lead to

$$-2k^2 = \pm\, \mathrm{Re}\, a_1 + \hat{\omega}\, \mathrm{Re}\, a_3$$

$$3k^2\lambda = \pm\, \mathrm{Im}\, a_1 + \hat{\omega}\, \mathrm{Im}\, a_3$$

$$\Longrightarrow \quad k^2(-2\, \mathrm{Im}\, a_3 - 3\lambda \mathrm{Re}\, a_3) = \pm\, (\mathrm{Re}\, a_1 \mathrm{Im}\, a_3 - \mathrm{Im}\, a_1\, \mathrm{Re}\, a_3)$$

These equations can be resolved w.r. to k and $\hat{\omega}$ if

(vi)' $\quad 2\mathrm{Im}\, a_3 + 3\lambda\, \mathrm{Re}\, a_3 \neq 0$.

For $\mathrm{Im}\, \tilde{a}_2 = 0$, once can again resolve (6.7) by $B = 0$, provided that

(vii)' If $\mathrm{Im}\, \tilde{a}_2 = 0$, then $\mathrm{Re}\, \tilde{a}_2$ is positive (note that this is the opposite of condition (vii)), and $\mathrm{Im}\, a_3 \neq 0$.

We thus find

<u>Proposition 6.3:</u>

If (vi)' or (vii)' respectively are satisfied, then the ansatz $\alpha_1 = re^{i\varphi}$, $r = C \tanh kx$, $\varphi' = B \tanh kx$ yields a solution $\alpha_1^*(x)$ to (6.5)

This solution α_1^* is an odd function of x and converges to periodic wave trains propagating in opposite directions as $x \to \pm\infty$. It agrees with the solution considered in [38] as a model for one-dimensional spiral waves. In that paper, (6.5) and its analogue in two space dimensions are considered for the special case $\mathrm{Re}\ a_3 = \mathrm{Im}\ a_1 = 0$. The solution $\alpha_1^*(x)$ is also the simplest of the solutions the existence of which has been proved in [23] for the case that $\mathrm{Im}\ \tilde{a}_2$ is small.

We now want to prove that, for sufficiently small ε, there exists a solution having the same characteristics as α_1^0 and α_1^*, respectively. We begin with the case of α_1^0. Linearizing (6.4) with respect to α and $\hat{\omega}$, we obtain the following inhomogeneous problem that has to be discussed

$$\beta_1' - \beta_2 = f_1$$

(6.8)

$$\beta_2' \mp a_1\beta_1 - \tilde{a}_2(\alpha_1^0)^2\bar{\beta}_1 - 2\tilde{a}_2\ \alpha_1^0\ \bar{\alpha}_1^0\ \beta_1 - a_3\hat{\omega}\beta_1 - a_3\Omega\alpha_1^0 = f_2$$

for which we write briefly.

$$L\beta + (0,-a_3\alpha_1^0)\Omega = f$$

As in chapter 3, we want to study properties of this linearized operator as a mapping from

$$R \times U_{m+1} = \{(\beta_1,\beta_2) \in C_b^{m+1}(R,C^2) \mid \lim_{x\to\infty} \beta_i^{(k)}(x), \lim_{x\to-\infty} \beta_i^{(k)}(x)$$
$$\text{exist for } 0\leq k\leq m+1,\ \beta_1(x) = \beta_1(-x),\ \beta_2(x) = -\beta_2(-x)\}$$

into

$$F_m = \{(f_1,f_2) \in C_b^m(R,C^2) \mid \lim_{x\to\infty} f_i^{(k)}, \lim_{x\to-\infty} f_i^{(k)}(x) \text{ exist for } 0\leq k\leq m,$$
$$f_1(x) = -f_1(-x),\ f_2(x) = f_2(-x)\}$$

The point $0 \in C^2$ is a saddle point for (6.4), which has two stable and two unstable directions. Hence the same arguments as in § 3 show that for any given $f \in F_m$ and $\Omega \in R$, we can find a bounded solution β on $[0,\infty)$, and two initial conditions at 0 are left arbitrary. These initial conditions can be matched by adding multiples of the solutions of the homogeneous problem, which are given by $(\alpha_1^{0'},\alpha_2^{0'})$ and $(i\alpha_1^0, i\alpha_2^0)$. As in § 3, the solution on $[0,\infty)$ extends to a solution on all of R, if $\beta_2(0)$ is zero.

As we can verify from (6.6), Re $\alpha_2^{0'}(0) \neq 0$, but $i\alpha_2^0(x)$ vanishes at zero. This means that by appropriate choice of the elements of the nullspace we can adjust one initial condition at $x = 0$, the other must be matched by appropriate choice of Ω. In other words, L (as an operator from U_{m+1} into F_m) has a one dimensional nullspace and the range has codimension one. We assume

(viii) $(0,-a_3\alpha_1^0)$ is not in the range of L

Now the same arguments as in § 4 lead to the result

Theorem 6.4:

Assume, conditions (i)-(viii) hold. Then for each ε in a neighborhood of 0 there exists $\hat{\omega} \in \mathbb{R}$ for which (6.4) has a one-parameter family of non-vanishing solutions, which are even in x and approach 0 as $x \to \pm\infty$. The solutions in this one parameter family differ from each other only by a shift in the time variable.

Remark:

Condition (viii) is of a "generic" type in the sense that it requires a certain quantity not to vanish. We have not succeeded in giving an explicit criterion, when (viii) is true. In one particular case, however, this can easily be seen, namely, assume that $\mathrm{Im}\ \tilde{a}_2 = 0$, and $\mathrm{Im}\ a_3 \neq 0$. Then it can be shown that the functional

$$(f_1,f_2) \to \mathrm{Im} \int_{-\infty}^{\infty} (-\alpha_2^0(x)f_1(x) + \alpha_1^0(x)f_2(x))dx$$

annihilates the range of L, and clearly, $(0,a_3\alpha_1^0)$ is not in the nullspace of this functional. It would be interesting to know whether there are parameter values for which (viii) is false.

For the case of the solution α_1^*, we have to introduce some new definitions of spaces, since we are now dealing with solutions approaching a periodic limit at infinity rather than a constant, and, moreover, the symmetry properties are different (α_1^* is odd rather than even), which means we shall have to make use of the reversibility under $\hat{R}$ rather than under R. Throughout the following discussion, we shall assume that $\operatorname{Im} \check{a}_2 \neq 0$, whence $B \neq 0$, and it is no restriction to assume $B > 0$, (otherwise change the signs of both B and k, which just corresponds to a symmetry transformation).

Definition 6.5:

Let σ be a positive real number. Then $Z^m_{\ell,\sigma}(\mathbb{R}^n)$ denotes the space of all functions $(u(x), v(x)) : \mathbb{R} \to \ell^1_m(\mathbb{R}^n) \times \ell^1_{m-1/2}(\mathbb{R}^n)$ such that the following hold:

1. u and v respectively are C^ℓ-functions of $x \in \mathbb{R}$
2. On $[0,\infty)$ u and v respectively can be represented as the sum of a 2π-periodic C^ℓ-function and the product of $e^{-\sigma x}$ with a function, whose first ℓ derivatives are continuous and bounded and converge to zero at infinity.
3. All odd Fourier components of u (with respect to t) are odd functions of x, all the even Fourier components are even functions of x, and vice versa for v.

$\hat{Z}^m_{\ell,\sigma}$ is defined in the same way, but condition 3 is to be taken the other way round. Clearly there is a natural choice for a norm in $Z^m_{\ell,\sigma}$, and we omit a detailed definition.

We fix a $\sigma > 0$, which is to be chosen sufficiently small, and we look for solutions to (6.4) in $Z^m_{\ell,\sigma}$ which lie in a neighborhood of $(\alpha_1^*(x), \alpha_2^*(x), \check{z} = 0)$. Since we wish to fix the period of the periodic part to 2π, a scaling factor γ must be introduced, for $\varepsilon = 0$ this factor equals B (i.e. in (6.4) we put $\check{x} = \gamma x$, but we shall again write x for $\check{x}$). Again we must discuss the linearization of (6.4) at $(\alpha_1^*, \alpha_2^*, 0)$. The third equation causes no problems, as can be seen from the results of § 4 and the fact that

$$(\varepsilon\gamma \frac{d}{dx} - \tilde{A}(0))\ (e^{-\sigma x}f(x)) = e^{-\sigma x}(\varepsilon\gamma \frac{d}{dx} - \varepsilon\gamma\sigma - \tilde{A}(0))f(x)$$

Let us now investigate the linearization of the first two equations of (6.4) at $(\alpha_1^*(x), \alpha_2^*(x)|$, i.e. we have to discuss the nullspace and codimension of the operator $D: D(D) \subset R^2 \times Z^m_{\ell,\sigma} \to \hat{Z}^m_{\ell,\sigma}$ given by

$$(6.9)\qquad \begin{aligned} D(\Omega.\Gamma,\beta_1,\beta_2) = (&\Gamma\alpha_1^{*'} + B\beta_1' - \beta_2,\ \Gamma\alpha_2^{*'} + B\beta_2' \mp a_1\beta_1 \\ &-a_3\hat{\omega}\beta_1 - 2\tilde{a}_2\ \alpha_1^*\overline{\alpha}_1^*\beta_1 - \tilde{a}_2\ (\alpha_1^*)^2\overline{\beta}_1 - a_3\Omega\alpha_1^*) \end{aligned}$$

Let $f = (f_1,f_2) \in \hat{Z}^m_{\ell,\sigma}$ be given and consider the problem $D(\Omega,\Gamma,\beta) = f$. Due to symmetry properties it is again enough to find solutions β on the half-line $[0,\infty)$, which satisfy $\beta_1(0) = 0$. We split β, f and α^* into the periodic and exponentially decaying parts, denoting them by indices p and d. The periodic part of (6.9) reads

$$(6.10)\qquad \begin{aligned} &B\beta_{1p}' - \beta_{2p} = f_{1p} - \Gamma\alpha_{1p}^{*'} =: \tilde{f}_{1p} \\ &B\beta_{2p}' \mp a_1\beta_{1p} - a_3\hat{\omega}\beta_{1p} - \tilde{a}_2(\alpha_{1p}^*)^2\overline{\beta}_{1p} - 2\tilde{a}_2|\alpha_{1p}^*|^2\beta_{1p} \\ &= f_{2p} - \Gamma\alpha_{2p}^{*'} + a_3\Omega\alpha_{1p}^* =: \tilde{f}_{2p} \end{aligned}$$

Here we have $\alpha_{1p}^* = \pm Ce^{\pm ix}e^{i\Phi}$, where C is the constant named so in proposition 6.3, the + or - sign agrees with the sign of k, and

$$\Phi = B\int_0^\infty (\tanh\,(kx) - 1)dx \qquad \text{for } k > 0$$

or

$$\Phi = B\int_0^\infty (\tanh kx + 1)dx \qquad \text{for } k < 0$$

Proposition 6.6:

The linear operator represented by the left side of (6.10) (in the space of 2π-periodic C^{ℓ}-functions) has a one dimensional nullspace spanned by $(\alpha_{1p}^{*\prime}, \alpha_{2p}^{*\prime})$ and codimension one. The functional annihilating the range is given by

$$(f_1,f_2) \to \operatorname{Im} \int_0^{2\pi} \frac{e^{\mp 2ix} e^{-2i\Phi}}{\check{a}_2} (\alpha_{2p}^{*} f_1 + \alpha_{1p}^{*} f_2)$$

Proof:

In (6.10) we put $\beta = e^{\pm ix} e^{i\Phi} \tilde{\beta}$, thus obtaining

$$\text{(6.11)} \qquad B\tilde{\beta}_{1p}' \pm iB\tilde{\beta}_{1p} - \tilde{\beta}_{2p} = e^{\mp ix} e^{-i\Phi} \tilde{f}_{1p}$$

$$B\tilde{\beta}_{2p}' \pm iB\tilde{\beta}_{2p} \mp a_1 \tilde{\beta}_{1p} - a_3 \hat{\omega} \tilde{\beta}_{1p} - \check{a}_2 |\alpha_{1p}^{*}|^2 \bar{\tilde{\beta}}_{1p} - 2\check{a}_2 |\alpha_{1p}^{*}|^2 \tilde{\beta}_{1p} = \tilde{f}_{2p} e^{\mp ix} e^{-i\Phi}$$

In order to find the nullspace, we make the ansatz

$$\tilde{\beta}_{1p} = \rho e^{inx} + \sigma e^{-inx}$$

This leads to the equations

$$((-n^2 \mp 2n)B^2 - \check{a}_2 C^2)\rho - \check{a}_2 C^2 \bar{\sigma} = 0$$

$$((-n^2 \pm 2n)B^2 - \check{a}_2 C^2)\sigma - \check{a}_2 C^2 \bar{\rho} = 0$$

Non-trivial solutions exist only if

$$((-n^2 \pm 2n)B^2 - \check{a}_2 C^2)((-n^2 \mp 2n)B^2 - \bar{\check{a}}_2 C^2) = \check{a}_2 \bar{\check{a}}_2 C^4$$

$$\Longleftrightarrow (n^4 - 4n^2)B^4 + 2n^2 B^2 C^2 \operatorname{Re}_2 \check{a} \pm 4iB^2 C^2 n \operatorname{Im}_2 \check{a} = 0$$

$$\Longrightarrow n = 0$$

From this it is easily seen that the nullspace is one dimensional. Since the resolvent is compact, so is the codimension. One can easily verify explicitly that the functional given above annihilates the range.

We now turn to the discussion of the decaying part. It is determined by the equations

$$
\begin{aligned}
& B\beta_{1d}' - \beta_{2d} = f_{1d} - \Gamma\alpha_{1d}^{*\prime} =: \hat{f}_{1d} \\
(6.12)\quad & B\beta_{2d}' \mp a_1\beta_{1d} - a_3\hat{\omega}\beta_{1d} - \tilde{a}_2(\alpha_1^*)^2\bar{\beta}_{1d} - 2\tilde{a}_2\alpha_1^*\bar{\alpha}_1^*\beta_{1d} \\
& \quad = f_{2d} - \Gamma\alpha_{2d}^{*\prime} + a_3\Omega\alpha_{1d}^* + \tilde{a}_2\,(\alpha_1^*)_d^2\bar{\beta}_{1p} + 2\tilde{a}_2(\alpha_1^*\bar{\alpha}_1^*)_d\,\beta_{1p} =: \hat{f}_{2d}
\end{aligned}
$$

<u>Proposition 6.7:</u>

For any given right side $(\hat{f}_{1d}, \hat{f}_{2d})$, equation (6.12) has a one-parameter family of solutions on $[0,\infty)$, which approach zero exponentially as $x \to \infty$.

<u>Proof:</u>

In the limit $x \to \infty$, all terms on the left of (6.12) that contain α_{1d}^* can be regarded as a perturbation (cf. § 3). Hence it suffices to show the proposition is true if we drop these terms. Again we substitute $\beta = e^{\pm ix}e^{i\Phi}\tilde{\beta}$, and we are left with the same left hand side as in (6.11), except that now we must look for solution decaying to zero at a rate of $e^{-\sigma x}$ rather than for periodic solutions.

Going through the same steps as we did following (6.11), we obtain the following equation for the characteristic exponents belonging to (6.10):

$$\lambda^4B^4 + 4B^4\lambda^2 - 2\lambda^2B^2C^2\mathrm{Re}\,\tilde{a}_2 \mp 4B^2C^2\lambda\,\mathrm{Im}\,\tilde{a}_2 = 0$$

One eigenvalue is $\lambda = 0$, and it is simple unless $\mathrm{Im}\,\tilde{a}_2 = 0$, which we have excluded. Using (6.7), we obtain for the remaining eigenvalues:

$$\lambda^3B^2 + 6\lambda B^2 - 4\lambda k^2 \pm 12kB = 0$$

which is solved by $\lambda = \mp 2\frac{k}{B}$, leaving the following equation for the remaining eigenvalues

$$\lambda^2B^2 \mp 2\lambda kB + 6B^2 = 0$$

$$\lambda = \frac{\pm 2kB \pm \sqrt{4k^2B^2 - 24B^4}}{2B^2}$$

The eigenvalue $\mp 2\frac{k}{B}$ is negative, and the last two eigenvalues have positive real parts. This yields the proposition.

Solutions on $[0,\infty)$ yield solutions to (6.12) on all of $\mathbb{R}$ if $\beta_1(0) = 0$. On $[0,\infty)$ we have two linearly independent solutions of the homogeneous problem, given by $(i\alpha_1^*, i\alpha_2^*)$ and $(\alpha_1^{*\prime}, \alpha_2^{*\prime})$. Since $\alpha_1^{*\prime}(0) \neq 0$, but $\alpha_1^*(0) = 0$, addition of a solution of the homogeneous problem can only be used to match one of the two initial conditions. Hence we see:

The operator $\tilde{D} = D|_{\Omega=\Gamma=0}$ has a one-dimensional nullspace and its range has codimension 2 (one coming from the condition $\beta_1(0) = 0$ and one from the codimension for the periodic part). Thus D is onto iff

(viii)' The vectors $(\alpha_1^{*\prime}, \alpha_2^{*\prime})$ and $(0, -a_3\alpha_1^{*\prime})$ span a complement to the range of D.

We do not know how to check (viii)' explicitly, but one can expect it to hold for almost all parameter values.

<u>Theorem 6.8:</u>

Suppose (i)-(v),(vi)',(viii)' hold and Im $\tilde{a}_2 \neq 0$. Then in a neighborhood of $\varepsilon = 0$ there exist $\gamma(\varepsilon)$, $\hat{\omega}(\varepsilon)$, for which (6.4) (with x scaled by γ) has a one-parameter family of non-periodic solution that lie in $Z^m_{k,\sigma}$. The solutions in this one-parameter family are again distinguished only by a time shift.

7. Stability

This chapter deals with the question whether the solution provided by theorem 6.4 is stable. We shall prove

Theorem 7.1:

If ν is real (e.g., if all diffusion coefficients are equal), $\mathrm{Im}\,\tilde{a}_2$ is small and $\mathrm{Re}\,\tilde{a}_2 < 0$, the solution given by theorem 6.4 is unstable.

Proof:

It is convenient to introduce artificially a second time variable τ. I. e., we proceed as follows: We seek solutions to (6.1), which have the form $u(t+\tau)$ and are periodic in t. The operator $\frac{\partial}{\partial t}$ must then be replaced by $\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}$. If we then go through the same transformations that led to (6.4), we arrive at the system

$$\alpha_1' = \alpha_2$$

$$(7.1) \qquad \alpha_2' = \pm\, a_1\alpha_1 + \tilde{a}_2\alpha_1^2\bar{\alpha}_1 + a_3\hat{\omega}\alpha_1 + \frac{1}{\varepsilon^2\nu}\left(b^*D, D^{-1}\left(\varepsilon\frac{\partial z}{\partial \tau} + \frac{\partial \alpha_1}{\partial \tau}\cdot y\right)\right) + O(|\varepsilon| + \|\tilde{z}\|)$$

$$\varepsilon\tilde{z}' = \tilde{A}(0)\tilde{z} + \left(1 - \frac{y\cdot b^*D}{(b^*D, y)}\right)D^{-1}\left(\frac{\partial z}{\partial \tau} + \frac{1}{\varepsilon}\frac{\partial \alpha_1}{\partial \tau}\cdot y\right) + O(\varepsilon)$$

Let us now assume that ν is real and $\tilde{a}_2$ is real (and negative). As a consequence, a_3 is imaginary, $a_1 + a_3\hat{\omega}$ is real and the plus sign must be chosen in front of a_1 (cf. (6.6)) for obtaining the solution $\alpha_1^0(x)$. We linearize (7.1) at the solution $\alpha_1 = \alpha_1^0(x)$, $\alpha_2 = \alpha_2^0(x)$, $\tilde{z} = 0$, and seek solutions of the linear equation which are proportional to $e^{\varepsilon^2\lambda\tau}$. For $\varepsilon = 0$ this yields the following eigenvalue problem for λ

$$\beta_1'' - a_1\beta_1 + 2\tilde{a}_2\beta_1(\alpha_1^0)^2 + \tilde{a}_2\bar{\beta}_1(\alpha_1^0)^2 + a_3\hat{\omega}\beta_1 + \frac{\lambda}{\nu}\beta_1 = 0$$

When β_1 is restricted to be real valued, we obtain

$$(7.2) \qquad \frac{\lambda}{\nu}\beta_1 = \beta_1'' - (a_1 + a_3\hat{\omega} + 3\tilde{a}_2(\alpha_1^0)^2)\beta_1$$

The derivative $\alpha_1^{0'}$ is a solution for $\lambda = 0$. The operator in L^2 represented by the right hand side of (7.2) is self-adjoint, and its eigenfunctions are critical points of the functional

$$\int_{-\infty}^{\infty} - \beta_1'^2 - (a_1 + a_3\hat{\omega} + 3\tilde{a}_2(\alpha_1^0)^2)\beta_1^2 dx$$

on the unit sphere of L^2. It is easy to prove that the maximizing function does not change its sign (this is a well-known principle in quantum mechanics) and thus cannot be $\alpha_1^{0'}$. Hence there is an eigenvalue $\lambda > 0$, and the linearized equation has an exponentially growing solution. Standard perturbation theory shows that this property is preserved under small perturbations in ε and $\tilde{a}_2$, which gives the theorem.

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Dynamical systems that are reversible in the sense of Moser are investigated and bifurcation of trajectories connecting saddle points from stationary solutions is studied. As an application, reaction-diffusion models in one space dimension are considered. These equations are studied in the neighborhood of a point, where the set of spatially homogeneous solutions displays a Hopf bifurcation. It is shown that from such a point branches of solutions bifurcate, which can be described as waves travelling

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ABSTRACT (continued)

to or from a center. These waves may be exponentially damped at infinity or not. They can be regarded as one-dimensional analogues of "target patterns" or "spiral waves".